Computing the price of the derivative that pays $(S_T^2 - K)^+$

Math 485

December 13, 2013

1 Goal:

To compute the price of a financial derivative that pays $(S_T^2 - K)^+$ at time T where S_t is a geometric BM:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

2 Derivation:

From the pricing formula:

$$V_0 = E(e^{-rT}(S_T^2 - K)^+).$$

Note that

$$S_T^2 = S_0^2 \exp((2r - \sigma^2)T + 2\sigma B_T).$$

We need to utilize the Black-Scholes formula, so we want to compare S_t^2 with a process with volatility 2σ. So we consider the process $\bar{S}_t$ where

$$d\bar{S}_t = r\bar{S}_t dt + 2\sigma \bar{S}_t dB_t.$$

That is

$$\bar{S}_t = \bar{S}_0 \exp((r - 2\sigma^2)t + 2\sigma B_t). \tag{1}$$

The Black-Scholes formula gives

$$E(e^{-rT}(\bar{S_T} - K)^+) = \bar{S}_0 N(\bar{d}_1) - Ke^{-rT}N(\bar{d}_2),$$

where

$$\bar{d}_1 = \frac{(r + 2\sigma^2)T - \log(\frac{K}{\bar{S}_0})}{2\sigma\sqrt{T}}$$
$$\bar{d}_2 = \frac{(r - 2\sigma^2)T - \log(\frac{K}{\bar{S}_0})}{2\sigma\sqrt{T}}$$

So in the original computation:

$$\begin{aligned} V_0 &= E(e^{-rT}(S_T^2 - K)^+) \\ &= E\left(e^{-rT}(S_0^2\exp((2r - \sigma^2)T + 2\sigma B_T) - K)^+\right) \\ &= e^{(r+\sigma^2)T}E\left(e^{-rT}(S_0^2\exp((r - 2\sigma^2)T + 2\sigma B_T) - \bar{K})^+\right), \end{aligned}$$

where $\bar{K} = \frac{K}{e^{rT+\sigma^2}}$. Now we have rewritten the formula in the form similar to (1) with $\bar{S}_0 = S_0^2$. Therefore, the conclusion is

$$V_0 = e^{(r+\sigma^2)T}[S_0^2 N(\bar{d}_1) - e^{-rT}\bar{K}N(\bar{d}_2)].$$